Appendix J

Dimensional and dimensionless quantities

J.1 Dimensional equations

If we change the magnitude of a unit of measurement by a certain factor, the magnitude of the measure will change as well by the inverse of that factor. For example, if the magnitude of the unit of length is reduced by a factor $L$, the measure of a length $l$ is multiplied by a factor $L$. This can be expressed by writing

$$[l] = L.$$

Analogously, the measure of an interval of time $t$ depends on the unitary interval of time in agreement with the relation

$$[t] = T.$$

For the derived quantity velocity $v$ its definition leads to

$$[v] = \frac{[l]}{[t]} = LT^{-1}.$$
In this case we say that the measure of a velocity is obtained as the ratio between the measure of a length $l$ and a time $t$. A derived quantity can be defined in terms of the measures of other derived and/or fundamental quantities. For example the acceleration $a$ is defined as the ratio between a velocity $v$ and a time $t$. Thus, we can write

$$[a] = \left[ \frac{v}{t} \right] = LT^{-2}.$$  

And so on for all the other derived quantities.

Since any derived quantity is introduced by a definition or by a physical law represented by a monomial algebraic relation, the measure of the quantity will be given by the product of particular integer or fractional powers of the other quantities. In other words, by changing the unit of measurement of the independent quantities, the measure of the dependent quantity will change according to a certain monomial relation. Let us consider some examples.

The momentum of inertia $I$ of a point mass of mass $m$ rotating around an axis at the distance $r$ is associated to the equation

$$[I] = [m][r]^2 = ML^2.$$  

This means that the measure of $I$ is given by the measure of the quantity $m$ times the square of the measure of the quantity $r$.

Once introduced the unit of measurement of the mass $M$, the unit of measurement of the distance from a straight line $B$ and the unit of measurement of the displacement in the direction of the force $S$, the moment $N$ of a force $F$ and the work $W$ are defined, respectively, by

$$[N] = [F][b] = MST^{-2}B = MSBT^{-2},$$ 

where $b$ is the arm of the force and

$$[W] = [F][s] = MST^{-2}S = MS^2T^{-2},$$ 

where $s$ is the displacement of the point of application of the force. Without the distinction between the two kind of length both quantities would have the same unit of measurement

$$[N] = [W] = ML^2T^{-2},$$ 

where $l$ represents whatever distance between two points. All the equations written in this section are called *dimensional equations*. 
Roughly speaking, the dimensional equations modify the concept of dimension, which so far has been presented as a qualitative concept, in a quantitative concept, providing it with a mathematical structure. Not only, but in this way we can arrange the various physical quantities on the basis of the complexity of their relation with the fundamental quantities.

J.2 Dimensionless quantities

So far we have only seen physical quantities where the change of the units of measurement lead to a change of the measure. But besides such quantities we can also have physical quantities that do not change as the units of measurement change.

The quantity \( \frac{I}{mr^2} \) with the definition given above is a quantity of this kind

\[
\left[ \frac{I}{mr^2} \right] = M^0 L^0 T^0.
\]

This relation is also written as

\[
\left[ \frac{I}{mr^2} \right] = [1].
\]

In other words the quantity \( \frac{I}{mr^2} \) has the dimension of a pure number. The numerical value of the measure is independent of the magnitude of the adopted units of measurement.

Since any algebraic expression can be rearranged so as to have at the second member the number 1, the construction of dimensionless quantities is very simple, not to say trivial. In general, a dimensionless quantity represents the fraction of a certain physical quantity with respect to another reference quantity of the same dimension.

The angle represents the only notable exception. When measured in radians, an angle \( \theta \) is defined as the ratio between the measure of the arc \( A \) (to be thought as the result of a rotation) and the measure of the radius \( R \) of the circle containing the arc (to be thought as the distance, deprived of direction, between the center of the circle and an arbitrary point of the circumference). We can thus write

\[
[\theta] = AR^{-1}.
\]

The dimension of an angle is not the same as that of a pure number, even if the unit of measurement of an angle is a pure number (the radian). However, by
considering both the arc and the radius as lengths of the same dimension, the angle becomes a nondimensional quantity.

Indeed, we might also easily measure the angles with reference to a full circle or half a full circle, that is, without any recourse to external units of measurements of length. But in this case the unit of measurement of the angle would no longer be a pure number, but it would have its own unit of measurement that would complicate the expressions of the various laws in which it might appear.

J.3 Equations for the units of measurement

If we indicate by \([q]_u\) the unit of measurement adopted for the physical quantity \(q\), within the framework of the International System we have

\[
[l]_u = \text{m}, \quad [t]_u = \text{s}, \quad [m]_u = \text{kg}, \quad [T]_u = \text{K}.
\]

For the derived quantities expressions similar to those adopted for the dimensional equations hold. For the velocity we have

\[
[v]_u = \frac{[l]_u}{[t]_u} = \text{m s}^{-1}.
\]

and so on for the other quantities.

Formally, we might decide to consider fundamental the unit of measurement of a derived quantity, and derived one of the fundamental units of the International System. It is a simple matter of algebraic elaborations. Recall that the choice of a set of fundamental quantities is a conventional choice. The dimensional equations, however, are valid independently of these choices.

Among the various laws of Physics there are some in which physical or universal constants appear. In the framework of the International System these quantities have a dimension. If we assume some of these quantities as fundamental quantities, then the units of measurement of all the other fundamental and/or derived quantities remain defined.

J.4 Physical systems of measurement

It is possible to free the definition of the fundamental units from the existence of physical samples. Therefore, the fundamental units can be evaluated as a function of a set of physical and universal constants.
For example, in the framework of Mechanics we might take the density and the
kinematic viscosity of the water at the triple point, and the universal gravitational
constant derivable from the Cavendish experiment. All these measures can be
performed in a laboratory with procedures of classical kind. If we would add the
specific heat at constant volume of the water at the triple point and recall that
the temperature has the dimension of an energy divided a specific heat, we might
also derive an physical unit of measurement for the temperature.

**Problem J.1** Determine the units of time, length and mass derived as a function of the
density $\rho$ and the kinematic density $\nu$ of the water at the triple point and of the universal
gravitational constant $G$.

**Answer.**


**Comment.** If we would lose the samples of the three quantities at the left member of the
equations, we may adopt other samples. Then, we might calculate in this new system
of measurement the values of the three constants on the right member. The values of
such constants in the International System would allow to evaluate the ratio between the
new samples and the old ones. The meter, the second and the kilogram could be easily
reconstructed.

The possibility to rely on measures of absolute kind would make superfluous
the presence of standards that, as time proceeds, might change, deteriorate or
gone lost. We should only never forget the values of the physical and universal
constants as computed within the framework of the International system. Clearly,
the accuracy of the reconstructed units would depend on the accuracy with which
the physical and universal constants are known at present.

Other systems of measurement might be obtained by choosing other basic
physical and/or universal constants. As we have already seen, the choice can be
extended to the whole realm of Physics.