Chapter 11

The Euler equations

11.1 Acceleration of a fluid parcel

In order to write the equations of motion for a fluid, we must place the observer in an inertial reference system. Then we have to apply the second principle of dynamics (Newton's second law) to its parcels. This requires a suitable expression for the acceleration of the parcels as a function of the velocity field. Bear in mind that when we speak of the acceleration of a parcel, we mean the time derivative of its velocity. But, according to the Eulerian approach, we need to express this quantity by means of a velocity field defined as a function of the time and space coordinates.

We are in the same situation encountered in section [5.2] for the evaluation of the time variation of a scalar property following the motion of the parcel. This relation was extended to an arbitrary vector quantity in (5.2). The only difference is that now the velocity field that advects the property and the property itself coincide. Thus, in this case (5.2) becomes

$$\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}. \tag{11.1}$$

It should be noted that the time derivative on the left is a *total derivative*, and represents the acceleration of the parcel, while the time derivative on the right is a *partial derivative*, that is, it represents the rate of change of the velocity in fixed points of space.

Let us analyze in greater detail the above expression. The acceleration of a parcel depends on two factors. The former, $\partial \mathbf{u}/\partial t$, also called *local acceleration*, represents the acceleration due to the fact that in a given point of space the velocity of the parcels passing through it can be either increasing or decreasing in time. The latter, $(\mathbf{u} \cdot \nabla)\mathbf{u}$, also called *advective term of the acceleration*, provides the acceleration due to the fact that a parcel can move from a region of low velocity to one of high velocity, or vice versa.

In a unidirectional flow that is uniform over a horizontal plane and possibly variable with the vertical coordinate, the advective component of the acceleration vanishes. In fact, the parcels during their motion do not undergo any change of velocity.

Let us see how this statement can be translated in mathematical terms. Let x be the direction of the motion. The three components of the advective term in the x-direction

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

all vanish: the first one vanishes because the velocity u does not vary with x, and the other two because both v and w are zero. Furthermore, the other two components of the acceleration in the y and z-directions vanish because both v and w are zero.

If the flow is also stationary, the local acceleration is zero as well, so that the total derivative of the velocity vanishes, in agreement with the fact that the parcels are not accelerated.

The second term of the acceleration is the source of the greatest mathematical difficulties encountered in the study of fluid mechanics. It is, in fact, a *nonlinear term*, indeed, a strongly nonlinear term. This prevents the use of the many mathematical tools available for the linear equations of all types, and forces the adoption and continuous research of new and increasingly advanced mathematical tools.

Problem 11.1 A rectilinear pipe of variable circular section $S = S(x)$ has a volume rate of flow Q. Evaluate the advective acceleration of the parcels lying along its central axis, assuming that their velocity is equal to the average velocity computed along a transversal section of the pipe.

Problem 11.2 Show that if $\mathbf{u} = \boldsymbol{\Omega} \times \mathbf{r}$, then

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \boldsymbol{\Omega} \times \mathbf{u}.$$

Provide the physical meaning of such an expression.

Solution. Let us orientate the coordinate axes in such a way that $\mathbf{\Omega} = \Omega\mathbf{k}$. Developing the given expression by components, we obtain

$$(\mathbf{u} \cdot \nabla)(\mathbf{\Omega} \times \mathbf{r}) = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)(-\Omega y\mathbf{i} + \Omega x\mathbf{j}) = -\Omega v\mathbf{i} + \Omega u\mathbf{j} = \mathbf{\Omega} \times \mathbf{u}.$$

The last term is nothing but the centripetal acceleration associated to the circular path of the parcels. In fact, if we decompose $\mathbf{r}$ in the sum of the two components $\mathbf{r}_{\parallel}$ and $\mathbf{r}_{\perp}$, respectively, parallel and perpendicular to the vector $\mathbf{\Omega}$, we have

$$\mathbf{\Omega} \times \mathbf{u} = \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = \Omega\mathbf{k} \times [\Omega\mathbf{k} \times (\mathbf{r}_{\parallel} + \mathbf{r}_{\perp})] = -\Omega^2\mathbf{r}_{\perp},$$

Problem 11.3 If we reverse the flows of the previous examples, does the sign of the advective component of the acceleration change?

Problem 11.4 On the basis of the examples presented, we see that for a general motion the advective acceleration is formed by at least two components: a tangential acceleration in the direction tangent to the trajectory of the parcel, and a centripetal acceleration in a direction normal to it, related to the curvature of the trajectory. Are there other terms in addition to these?

Now that we have an expression for the parcel acceleration, we can write the *momentum equation*, i.e., the equation which provides an expression of the parcel acceleration. By applying the second principle of dynamics one has

$$\rho\frac{d\mathbf{u}}{dt}\,\delta x\delta y\delta z = \rho\,\mathbf{F}\,\delta x\delta y\delta z. \tag{11.2}$$

Here, ρ is the fluid density, $\delta x\delta y\delta z$ the volume of the fluid parcel, $\delta m = \rho\,\delta x\delta y\delta z$ its mass and $\mathbf{F}$ the force per unit mass exerted on it.

The gravity force is clearly unaltered by the motion. Thus our attention must be concentrated on the surface stresses. We have already studied the behavior of the pressure necessary for an hydrostatic equilibrium. Any change in the values of the pressure with respect to its hydrostatic distribution will give rise to some kind of motion.

11.2 The Euler equations

The only forces present in a fluid at rest are the pressure forces. It is difficult to think that, as soon as the fluid moves, such forces disappear or are deeply changed. Thus, it seems reasonable to assume that the structure of the forces on the infinitesimal parcel remains the same found in the hydrostatic case also for a fluid in motion. In other words, we may assume that the stresses between parcels are always normal to their surfaces of separation and independent of their orientation. The symbol used to denote the scalar quantity defining the magnitude of this forces will remain the same, along with the name, pressure (or more exactly *dynamic pressure*, to point out that now the pressure varies not only in space, but also in time).

In this way, the expression of the force would remain the same already found in (10.1) as a function of the hydrostatic pressure

$$\mathbf{F} = -\frac{\nabla p}{\rho} + \mathbf{g}.$$

By inserting such expression in (11.2) along with expression (11.1) for the acceleration of the parcel, we have

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla p}{\rho} + \mathbf{g}. \tag{11.3}$$

These equations are known as the *Euler equations*, after the name of the author that derived them for the first time. Written in component form, they become

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{11.4}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{11.5}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g. \tag{11.6}$$

Problem 11.5 State the condition for the validity of the hydrostatic balance.

Solution. When $dw/dt = 0$, i.e., when the vertical acceleration vanishes, (11.6) reduces to (10.3). In particular, the hydrostatic balance holds for any purely horizontal motion.

It turns out that these equations can describe the structure of the motion only in some circumstances. In many cases the solution is a good approximation of the real flow only in certain regions of space, but not in others. In other cases the real motion is completely different.

The simplest hypothesis the we can formulate is that the pressure forces are not the only forces present in a fluid. Thus, other forces must exist that certain kinds of motions might underline.

11.3 Properties of the flows governed by the Euler equations

The Euler equations state that when pressure gradients develop inside a fluid in addition to those relative to the hydrostatic equilibrium, then the fluid parcels react by accelerating in the opposite direction, the magnitude of the accelerations being proportional to the magnitude of the pressure gradients.

Such pressure gradients in the interior of the fluid can be produced by normal stresses exerted along the external boundaries of the fluid or, in the case of a liquid, by the displacement of the free surface from its equilibrium position.

For example, when we shake a bowl full of water, we give rise to a deformation of the free surface with respect to the equilibrium condition represented by a horizontal plane. This generates the pressure gradients along the horizontal surfaces that cause the motion of the fluid.

From a theoretical point of view, it is particularly meaningful to consider the plane-parallel flows, in spite of their triviality. The vertical component of the Euler equations (11.6) reduces to the hydrostatic equation (10.3). The horizontal component (11.4) of the Euler equations in the direction of the motion (the other equation (11.5) is identically satisfied term by term) then state that each layer of fluid moves as a rigid body, where the acceleration of the parcels is proportional to the pressure gradient in the direction of the motion at the same level, so that the various layers, whose density might also be different, move independently from each other.

Problem 11.6 Consider a unidirectional flow, that is uniform on the horizontal plane. Find a relationship between the velocity and the pressure gradient, assuming that both fields vary sinusoidally in time.

Solution. Let us orientate the flow in the x-direction. The x-component of the Euler equations yields

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

since in this case the advective term of the acceleration vanishes. Let the velocity field vary in time as

$$u = U\cos\omega t.$$

A substitution of this expression in the previous equation leads to

$$-\omega\, U\sin\omega t = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

from which

$$\frac{\partial p}{\partial x} = \rho\,\omega\, U\sin\omega t.$$

Thus, the pressure is out of phase by 90° with respect to the velocity field.

Comment. This solution can be easily verified in shallow water waves, and represents one of the many experimental proofs of the validity (in the appropriate circumstances) of the Euler equations.

11.4 The ideal fluid

As already said, since the Euler equations are not able to describe every kind of motion, other forces must be present in a fluid, that only in particular circumstances, or in particular regions of the space can be negligible. The pressure forces do not account for the whole motion, even if they are always very important and meaningful.

For this reason the concept of *ideal, or perfect fluid* has been introduced. The solutions found by means of the Euler equations, are thought as the solutions valid for a particular kind of fluid, that, however, does not exist in nature. Thus we can say something like *the fluid behaves as if it were perfect*, in order to make a simple and immediate reference to the structure of the internal forces acting inside it.

Indeed, when we speak of a perfect fluid we refer not to a property of the fluid, but to a property of the motion. The same fluid can be considered either ideal or not, depending on the case.

11.5 Historical notes and essential bibliography

The continuity and Euler equations were derived by Leonhard Euler in 1755 [16] both in Lagrangian and in Eulerian form. He had to devise the mathematical tools suited to deal with a continuous medium, that is, the partial derivatives and the vector differential operators, by introducing the concept of parcel. This was a fundamental progress with respect to the laws introduced by Galileo and Newton, referring to simple point-masses.

A description of the Lagrangian approach is contained in Lamb [30], along with a significant number of solutions of the Euler equations already found at the time of the publication of the last edition (the sixth) of the book, in 1932. Other classical solutions are shown in Milne-Thomson [33].