Chapter 24

Blasius solution

24.1 Boundary layer over a semi-infinite flat plate

Let us consider a uniform and stationary flow impinging tangentially upon a vertical flat plate of semi-infinite length (Fig. 22.1). Furthermore, assume that the fluid is moving at the constant velocity U in the x direction in the half-space $x < 0$ and that the flat plate is placed along the half-plane $y = 0$, $x > 0$ with which the previous flow interacts.

On the basis of the solution for an impulsive flow over an infinite plate we can suppose that the transition of the velocity field to a zero value along the plate can take place in a thin boundary layer of thickness much smaller than the distance from the origin of the plate.

Then, we can consider that the plate from a point in the vicinity of the boundary looks like as if it were extended from $-\infty$ to $+\infty$. Therefore, we can suppose that the horizontal velocity will depend on a dimensionless quantity similar to (22.3), but with the substitution

$$t = \frac{x}{U},$$

which represents the time during which the flow has felt the presence of the

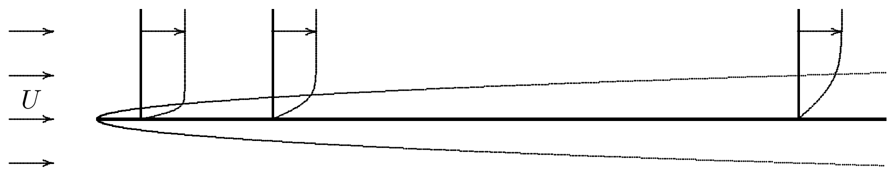

Fig. 24.1: The thickness of the boundary layer along a semi-infinite plate increases with the square of the distance from the edge. For each quadrupication of the distance from the edge of the plate we have to double the distance from the plate in order to find the same velocity. The vertical profile of the velocity, shown in Fig. 22.2 is similar, but not equal, to the profile relative to an impulsive flow over an infinite plate shown in Fig. 22.1.

plate. Thus, we can suppose that the solution will depend on the nondimensional parameter

$$\eta = y \left(\frac{U}{\nu x} \right)^{1/2} . \tag{24.1}$$

Here the factor 2 has been omitted because it is no longer useful for the simplification of the subsequent calculations.

Let us further suppose that the behavior of the flow near the edge of the plate, were the previous arguments are no longer valid, is irrelevant for the behavior of the flow far from it.

Indeed, the whole problem depends on two nondimensional parameters, the second of which might be y/x. We assume that far from the origin this last parameter is uninfluential.

The sum of these three hypothesis allow us to solve the problem far from the edge with a very good approximation.

24.2 The equations of motion

If we consider a homogeneous fluid (which is not particularly restrictive within the thin boundary layer), according to the equations of the horizontal momentum for a stationary flow derivable from (18.11) and the two-dimensional continuity equation derivable from (5.3) become

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \tag{24.2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{24.3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{24.4}$$

The continuity equation (22.4) can provide information about the magnitude of the transversal velocity. Let us consider a rectangle including the boundary layer with a long side resting on the plate a short side in the transversal direction. We see immediately that the flow entering the rectangle in the horizontal direction from the side closer to the beginning of the plate is greater than the flow exiting from the other short side. In fact, as the flow moves far from the origin, the plate manifests its presence by reducing the mean velocity of the flow parallel to it inside the rectangle.

For continuity the transversal velocities along the long side of the rectangle not in contact with the plate must give rise to an outgoing flow compensating the reduction of the flow in the parallel direction. The transversal velocities start from zero at the plate and increase with the distance from it, reaching the maximum value along the farthest long side of the rectangle.

The global flow is thus moved away from the plate, even if it remains essentially horizontal. This effect, which is of secondary importance, was completely absent in the impulsive flow. The ratio between the mean transversal velocity and the mean velocity parallel to the plate is of the order of the ratio between the thickness of the boundary layer and the distance from the origin.

A comparison of the terms depending on the velocity in (22.2) and in (22.3) shows that the latter are smaller than the former. The pressure term $-(\partial p/\partial y)/\rho$ must be of the same order of magnitude of the other terms of (22.3). But far from the plate the pressure is constant. Hence, within the boundary layer, if the variations of the pressure are small in the transversal direction y, they must be small in the parallel direction x as well. It follows that the pressure term in (22.2) can be neglected.

The system formed by (22.2) and (22.4) is a system of two equations in two unknowns. It can be solved providing as a result the components of the horizontal velocity. Then, (22.3) can be used to derive the small perturbations in the pressure field.

On the other hand the second derivative of the parallel velocity in the transversal direction $\partial^2 u/\partial y^2$ far from the origin and close to the boundary of the plate is clearly much larger than the second derivative in the parallel direction $\partial^2 u/\partial x^2$, so that the latter term can be omitted as well.

This last hypothesis is surely not valid near the edge of the plate where the velocity at $y = 0$ changes sharply from U to 0 when x varies from $0-$ to $0+$.

Therefore, the equations of motion can be reduced in the form

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{24.5}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{24.6}$$

Further simplifications are impossible. At least one of the two nonlinear terms of (22.5) must be important, in order to balance the viscous term, which obviously cannot be negligible in the thin boundary layer. The problem in thus intrinsically nonlinear.

Indeed, it is one of the simplest nonlinear problems encountered in fluid dynamics.

The boundary conditions are

$$u = 0, \quad \text{for } y = 0, \tag{24.7}$$

$$v = 0, \quad \text{for } y = 0, \tag{24.8}$$

$$u \to U, \quad \text{for } y \to \infty. \tag{24.9}$$

24.3 The Blasius equation

The second basic hypothesis discussed in section [22.1] can be summarize by the expression

$$u = Ug(\eta), \tag{24.10}$$

where η because of (22.1) can be written as

$$\eta = \frac{y}{\delta(x)} \tag{24.11}$$

with

$$\delta(x) = \left(\frac{\nu x}{U}\right)^{1/2}, \tag{24.12}$$

so that

$$\frac{\partial \eta}{\partial x} = -\frac{\eta}{\delta}\frac{\partial \delta}{\partial x}, \tag{24.13}$$

$$\frac{\partial \eta}{\partial y} = \frac{1}{\delta}. \tag{24.14}$$

The continuity equation (22.6) allows us to write the two components of the velocity by means of a streamfunction as in (E.1)–(E.2)

$$u = +\frac{\partial \psi}{\partial y}, \tag{24.15}$$

$$v = -\frac{\partial \psi}{\partial x}. \tag{24.16}$$

The momentum equation (22.5) becomes

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3}. \tag{24.17}$$

On the other hand the streamfunction can be expressed in the following form

$$\psi = \int_0^y u\, dy = \delta \int_0^\eta u\, d\eta = \delta \int_0^\eta U g(\eta)\, d\eta = U\delta f(\eta), \tag{24.18}$$

where

$$g(\eta) = \frac{df}{d\eta}. \tag{24.19}$$

From (22.16), (22.18) and (22.13) it follows that

$$v = -\frac{\partial \psi}{\partial x} = -U\left(f\frac{d\delta}{dx} + \delta\frac{\partial f}{\partial x}\right) = -U\frac{d\delta}{dx}\left(f - \eta\frac{df}{d\eta}\right). \tag{24.20}$$

From (22.15) and (22.18) (or (22.10) and (22.19) we have

$$u = \frac{\partial \psi}{\partial y} = U\frac{df}{d\eta}. \tag{24.21}$$

The derivatives of higher order of ψ can be obtained similarly

$$\frac{\partial^2 \psi}{\partial x \partial y} = U\frac{d\delta}{dx}\frac{\partial}{\partial y}\left(f - \eta\frac{df}{d\eta}\right) = -U\frac{\eta}{\delta}\frac{d^2 f}{d\eta^2}\frac{d\delta}{dx}, \tag{24.22}$$

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{U}{\delta}\frac{d^2 f}{d\eta^2}, \tag{24.23}$$

$$\frac{\partial^3 \psi}{\partial y^3} = \frac{U}{\delta^2}\frac{d^3 f}{d\eta^3}. \tag{24.24}$$

Inserting (22.20)–(22.24) into (22.17), and noting that from (22.12)

$$\frac{U}{\nu}\delta\frac{d\delta}{dx} = \frac{1}{2},$$

we arrive at

$$\frac{1}{2}f\frac{d^2f}{d\eta^2} + \frac{d^3f}{d\eta^3} = 0. \tag{24.25}$$

Thus, we have traced back our system of partial differential equations (22.5)–(22.6) to an ordinary equation, although of the third order. This equation is nonlinear, as expected. It is referred to as the *Blasius equation* after the name of the author that discovered it. The boundary conditions to satisfy are

$$f = 0, \quad \text{for } \eta = 0,, \tag{24.26}$$

$$\frac{df}{d\eta} = 0, \quad \text{for } \eta = 0,, \tag{24.27}$$

$$\frac{df}{d\eta} \to 1, \quad \text{for } \eta \to \infty. \tag{24.28}$$

The last condition (22.28) is (22.9) worked out through (22.10) and (22.19). Condition (22.27) is a reworking of (22.7), always on the basis of (22.10) and (22.19). Condition (22.26), instead, derives from (22.20), because for $\eta = 0$ the first and the last terms vanish. This second term vanishes both because $\eta = 0$ and because of (22.27).

24.4 Numerical solution of the Blasius equation

An analytical solution in closed form uniformly convergent in the whole domain is not available. However, the equation can be solved numerically with the wanted accuracy (Fig. 22.2).

To say that the solution depends on the ratio between the y-coordinate and the root square of the x-coordinate implies that the thickness of the boundary layer increases with x (Fig. 22.1). If we define such a thickness as the distance at which the velocity differs for less than 1% with respect to the velocity U at infinity, then we have

$$\delta_l = 4.9\left(\frac{\nu x}{U}\right)^{1/2},$$

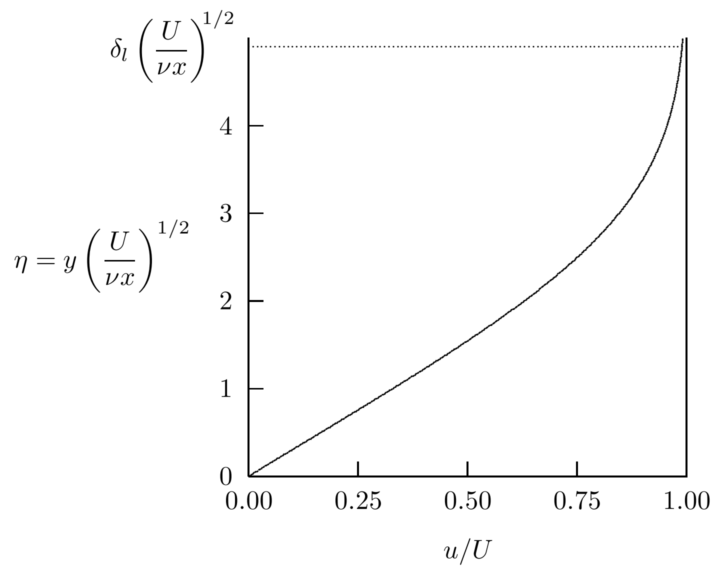

Fig. 24.2: Blasius solution for a semi-infinite plate. The horizontal dotted line indicates the thickness of the boundary layer, where the velocity is equal to 99% of the interior velocity.

corresponding to $\eta = 4.9$. This confirms one of the basic hypotheses, i.e., that the thickness of the boundary layer increases very slowly.

Problem 24.1 Establish the order of magnitude the terms present in the momentum equation (22.5).

Comment. When $y = 0$ all the terms are evidently zero. Using the Blasius solution we see that they increase progressively as we move away from the plate. This means that it is impossible to assign an order of magnitude to the various terms of the equation valid everywhere. The structure of the advective terms is of the kind $-a$, and $a - b$, with b slightly greater than a and $a > a - b$. Although the second term is smaller than the first one, to neglect it would only worsen the solution without any gain.

Problem 24.2 Evaluate the derivative of v with respect to y for $y = 0$.

Solution. From (22.20) we obtain

$$\frac{\partial v}{\partial y} = -\frac{U}{\delta}\frac{d\delta}{dx}\frac{\partial}{\partial \eta}\left(f - \eta\frac{df}{d\eta}\right) = \frac{U}{\delta}\frac{d\delta}{dx}\eta\frac{d^2 f}{d\eta^2}.$$

Therefore, for $\eta = 0$ we have $\partial v/\partial y = 0$. Thus the transversal velocity increases very slowly in the vicinity of the plate.

Problem 24.3 Evaluate the stress per unit area at the surface of the semi-infinite plate.

Solution. From the definition of surface stress

$$T_s = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu U \left.\frac{d^2 f}{d\eta^2}\right|_{\eta=0} \frac{\partial \eta}{\partial y} = \mu \frac{U}{\delta} \left.\frac{d^2 f}{d\eta^2}\right|_{\eta=0} \simeq 0.332\, \mu \frac{U}{\delta}.$$

Comment. Since $\delta \propto x^{1/2}$, then $T_s \propto x^{-1/2}$.

The velocity fields around a finite plate can be assumed as essentially similar to those encountered in a plate of semi-infinite length. This depends on the fact that the equations of motion are of parabolic type, so that what happens up to a certain distance does not depend on what happens at a greater distance. Thus, the total stress on a finite plate can be calculated integrating the tangential stress of the Blasius solution over the sole length of the plate. This theoretical value turns out to be in agreement with the experimental results.

Problem 24.4 Evaluate the total force F on the two faces of a plate of length l.

Solution. We can assume that the solution of the problem is the same we have found for a semi-infinite plate between $x = 0$ and $x = l$. From the solution of problem [22.3] and (22.12) we obtain

$$F = 2 \times 0.332\, \rho U^{3/2} \nu^{1/2} \int_0^l x^{-1/2} dx = 1.328 \left(\rho \mu U^3 l\right)^{1/2}.$$

Indeed, the Blasius solution breaks down at a certain distance from the origin. It can be applied successfully to describe the field around a plate of finite length or in the first part of a longer plate (the semi-infinite length being in any case an unrealizable abstraction).

24.5 Flow between two parallel plates

Let us consider the stationary flow between two semi-infinite flat plates placed in the positive half space. Let us suppose that the flow at the intake is uniform. What we see is the formation of a boundary layer of increasing thickness with the distance from the intake. The velocity profile, which is constant at the intake, at a short distance is still almost constant in the central region between the two boundary layers. Then, at a greater distance, it becomes more and more rounded because of the contribution of the transversal velocities generated within the boundary layers, which, for continuity, give rise to an intensification of the longitudinal velocity. At a certain distance the two boundary layers merge in a single viscous flow with a velocity profile approaching the parabolic profile of the Poiseuille flow between two infinite parallel plates considered in section [13.1].

Therefore, the Poiseuille solution for two infinite plates is no longer a simple theoretical solution of a rather unrealistic character, but the asymptotic solution observable in a flow between two plates of finite length. The dynamics of a pipe of circular section of semi-infinite or finite length is similar.

It is possible to prove that the parabolic profile is reached at a distance equal to $0.08\,\Re$, where the Reynolds number is referred to the half-distance between the two plates. Thus, for $\Re = 1\,000$ such ratio is equal to 80, and grows to 200 for $\Re = 2\,500$. Similar results hold for a pipe of circular section. The intake effects extend to a very large distance in a viscous laminar flow.

24.6 Nature of the Blasius solution

The Blasius solution is based, in the present derivation, on three hypothesis suggested by the observation or experimentally verifiable. The transition of the velocity field to zero occurs in a layer so thin that it cannot be easily seen. We have the impression that both in air and water the flow slides over the solid surfaces without friction. Only the study of the laminar flows in capillary vessels suggests that the boundary condition for a viscous fluid must be given by a zero velocity. The extension of this law to every flow is not trivial.

The impulsive flow suggests the possibility of a similar solution far from the edge of the plate. The assumption that the border of the plate cannot generate effects able to propagate along the plate can be verified only *a posteriori*.

It is possible to find more accurate solutions of the boundary layer equations by a series expansion approach. But the starting point of the mathematical solution relies on an experimental support.

24.7 Historical notes and essential bibliography

The laminar boundary layer was discovered by Ludwig Prandtl in 1904, in the context of complicated flows around three-dimensional bodies [43]. The solution for the semi-infinite plate was obtained by Heinrich Blasius [4] in 1908.