Appendix E

Two-dimensional differential operators

E.1 The del-operator in two dimensions

When we deal with scalar fields that do not depend on the third coordinate or with vector fields in which the third component is zero and the other two depend only on the first two coordinates, then it is expedient to introduce the horizontal del-operator

$$\nabla_H = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right).$$

By means of this operator we can construct the horizontal gradient of a two-dimensional scalar field $\phi = \phi(x, y)$

$$\mathbf{u}_H = \nabla_H \phi,$$

and the divergence of a two-dimensional vector field $\mathbf{u}_H = (u_x, u_y)$

$$\xi_H = \nabla_H \cdot \mathbf{u}_H = (\partial_x, \partial_y) \cdot (u_x, u_y) = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}.$$

Analogously to the three-dimensional case, we introduce the two-dimensional Laplace operator

$$\nabla_H^2 \phi = \nabla_H \cdot \nabla_H \phi = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \cdot \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}.$$

But for the curl is quite another matter. The curl, in fact, is an operator essentially three-dimensional. When applied to a two-dimensional vector field defined in a certain plane, it gives rise to a vector that does not belong to the same plane. This happens because it represents the velocity of rotation of the parcels (see problem [D.5]). If the motion occurs in a horizontal plane, it is clear that the angular velocity of the parcels must be vertical. So, we can define the vertical component of the curl as

$$\zeta = \mathbf{k} \cdot \nabla \times \mathbf{u}_H = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}.$$

Similarly, the curl of a vector field defined in a vertical plane is a horizontal vector.

Consider now a vertical vector field

$$\boldsymbol{\psi} = (0, 0, \psi),$$

with $\psi = \psi(x, y)$, and carry out its curl

$$\mathbf{u}_H = \nabla \times \mathbf{k}\psi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & \psi \end{vmatrix} = \frac{\partial \psi}{\partial y}\mathbf{i} - \frac{\partial \psi}{\partial x}\mathbf{j} + 0\mathbf{k}.$$

More explicitly, the components of $\mathbf{u}_H$ are

$$u_x = +\frac{\partial \psi}{\partial y}, \tag{E.1}$$

$$u_y = -\frac{\partial \psi}{\partial x}. \tag{E.2}$$

This field is clearly not divergent, since the mixed derivatives of a regular function are equal, regardless of the order in which they are performed. The function ψ is called *(scalar) streamfunction.*

Let us analyze in depth the meaning of the streamfunction. First of all observe that, according to problem [D.11.3], it is

$$\mathbf{u}_H = \nabla \times \psi\mathbf{k} = -\mathbf{k} \times \nabla_H \psi. \tag{E.3}$$

Hence, the vector $\mathbf{u}_H$ is rotated clockwise by 90° with respect to the gradient of ψ. This means that the isolines of ψ are *streamlines*, that is, lines tangent in every point to the vector field. (The name derives from the meaning of the line when the vector field is the velocity.) Furthermore, the magnitude of the gradient of ψ is equal to the magnitude of $\mathbf{u}_H$.

From (E.3) it also results that the cross product of the unit vertical vector and the gradient of some scalar field can be seen as the curl of an appropriate vector field.

E.2 Basic properties of two-dimensional operators

Two-dimensional versions of the relationships (D.3) and (D.4) hold

$$\mathbf{k} \cdot (\nabla \times \nabla_H \phi) = 0,$$
$$\nabla_H \cdot (\nabla \times \psi \mathbf{k}) = 0.$$

Moreover, if

$$\mathbf{k} \cdot \nabla \times \mathbf{u}_H = 0,$$

then a scalar field $\phi = \phi(x, y)$ exists such that

$$\mathbf{u}_H = \nabla_H \phi.$$

Similarly, if

$$\nabla_H \cdot \mathbf{u}_H = 0,$$

then a scalar function $\psi = \psi(x, y)$ exists such that

$$\mathbf{u}_H = \nabla \times \psi \mathbf{k}.$$

These last two theorems are important because they permit the representation of a two-dimensional vector field in terms of a simple scalar field.

Problem E.1 Demonstrate that in two dimensions

$$\nabla_H \cdot \mathbf{r} = 2, \qquad \nabla_H \times \mathbf{r} = 0, \qquad (\boldsymbol{\Omega} \cdot \nabla)\mathbf{r} = \boldsymbol{\Omega}_H,$$

where $\boldsymbol{\Omega}_H$ is the horizontal component of $\boldsymbol{\Omega}$.

Comment. Note the similarities and differences between this problem and its three-dimensional version [D.10].

Problem E.2 Show that if $\mathbf{\Omega} = \Omega(r)\,\mathbf{k}$ and $\mathbf{u}_H = \mathbf{\Omega} \times \mathbf{r} = \mathbf{\Omega} \times \mathbf{r}_H$, then

$$\nabla_H \cdot \mathbf{u}_H = 0.$$

Solution. The components of $\mathbf{u}_H$ are $u = -y\,\Omega(r)$ and $v = x\,\Omega(r)$, so that

$$\frac{\partial u}{\partial x} = -y \frac{\partial \Omega}{\partial r} \frac{\partial r}{\partial x} = -\frac{xy}{r} \frac{\partial \Omega}{\partial r}.$$

and

$$\frac{\partial v}{\partial y} = \frac{xy}{r} \frac{\partial \Omega}{\partial r}.$$

Since the two terms are opposite, the final result follows.

Comment. This is an extension of a result of problem [D.5] to the case of a circular motion of arbitrary radial structure.

Problem E.3 Demonstrate that

$$\nabla_H^2 \log r = 0,$$

where $r^2 = x^2 + y^2 > 0$.

Problem E.4 Demonstrate that for a quantity $u = u(r)$, with $r^2 = x^2 + y^2$, we have

$$\nabla_H^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right). \tag{E.4}$$

Comment. The proof is similar to that of problem [D.14] relative to the three-dimensional case.

E.3 Two-dimensional potential fields

If the field is not only solenoidal, but also irrotational, then the lines along which the potential and the streamfunction are constant intersect each other normally. In fact, (E.3) states that the gradient of ϕ is orthogonal to the gradient of ψ. The potential field satisfies the two-dimensional Laplace equation

$$\nabla_H \cdot \mathbf{u}_H = \nabla_H \cdot \nabla_H \phi = \nabla_H^2 \phi.$$

Analogously, since the vertical component of the curl is zero, the streamfunction satisfies the Laplace equation as well

$$\mathbf{k} \cdot \nabla \times \mathbf{u}_H = \frac{\partial}{\partial x} \left(-\frac{\partial \psi}{\partial x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial \psi}{\partial y} \right) = -\nabla_H^2 \psi = 0.$$

Equation (E.3) written component by component reads

$$\frac{\partial \phi}{\partial x} = +\frac{\partial \psi}{\partial y},$$
$$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}.$$

These equations can be interpreted as the Cauchy-Riemann conditions necessary and sufficient for the differentiation of a function of complex variable.

Let us introduce the complex variable

$$z = x + iy.$$

Then, the real and imaginary part of every complex function

$$f(z) = f(x + iy) = \phi(x, y) + i\psi(x, y)$$

that is differentiable with respect to z represent the potential and the stream functions of a possible irrotational and solenoidal vector field. The harmonic conditions for ϕ and ψ can be derived from the non-divergent and irrotational properties of the field.

E.4 Historical notes and essential bibliography

A detailed and elementary treatment of the mathematical tools considered so far and addressed to the study of a geophysical fluid is present in Riegel [47], chapters 2 and 3. At a slightly higher level we have the classical book by Kundu [29], chapter 2. Tensors are also discussed in Brown [6], chapter 4.